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TRIPLE DIRICHLET AVERAGE OF NEW GENERALIZATION OF GENERALIZED M-SERIES $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{z})$ AND FRACTIONAL DERIVATIVE

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ABSTRACT

The aim of present paper to establish some results of Triple Dirichlet average of ***New generalization of Generalized M-series*** $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{z})$, using fractional derivative.

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INTRODUCTION

Carlson [1-5] has defined Dirichlet average of functions which represents certain type of integral average with respect to Dirichlet measure. He showed that various important special functions can be derived as Dirichlet averages for the ordinary simple functions likex^t,e^x etc. He has also pointed out [3] that the hidden symmetry of all special functions which provided their various transformations can be obtained by averaging x^n,e^x etc. Thus he established a unique process towards the unification of special functions by averaging a limited number of ordinary functions. Almost all known special functions and their well-known properties have been derived by this process.

Recently, Gupta and Agarwal [9, 10] found that averaging process is not altogether new but directly connected with the old theory of fractional derivative. Carlson overlooked this connection whereas he has applied fractional derivative in so many cases during his entire work. Deora and Banerji [6] have found the double Dirichlet average of e^x by using fractional derivatives and they have also found the Triple Dirichlet Average of x^t by using fractional derivatives [7].

In the present paper the Double Dirichlet average of new generalization of Generalized M-series has been obtained.

DEFINITIONS

Some definitions which are necessary in the preparation of this paper.

Standard Simplex in $R^n, n \geq 1$

The standard simplex in R^n, $n \geq 1$ by [1, p.62].

$$E = E_n = \{S(u_{1,}u_2, \ldots \ldots \ldots u_n) \; : \; u_1 \geq 0, \ldots \ldots \ldots . u_n \geq 0, \; u_1 + u_2 + \cdots \ldots \ldots + u_n \leq 1\}$$

Dirichlet measure

Let $b \in C^k, k \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b is defined by $E[1]$.

$$d\mu_b(u) = \frac{1}{B(b)} u_1^{b_1-1} \ldots \ldots \ldots \ldots \ldots u_{k-1}^{b_{k-1}-1}(1 - u_1 - \cdots \ldots \ldots \ldots - u_{k-1}) b_k^{-1} du_1 \ldots \ldots \ldots \ldots . du_{k-1}$$

Will be called a Dirichlet measure.
Here

$$B(b) = B(b1, \ldots \ldots \ldots . bk) = \frac{\Gamma(b_1) \ldots \ldots \ldots \ldots \ldots . \Gamma(b_k)}{\Gamma(b_1 + \cdots \ldots \ldots . + b_k)},$$

$$C_> = \{z \in z : z \neq 0, |ph\, z| < {}^{\pi}/_{2}\},$$

Open right half plane and $C_>$k is the k^{th} Cartesian power of $C_>$

Dirichlet Average[1, p.75]

Let Ω be the convex set in $C_>$, let $z = (z_1, \dots\dots\dots, z_k) \in \Omega^{\mathrm{k}}, \mathrm{k} \geq 2$ and let $u.z$ be a convex combination of $z_1, \dots\dots\dots, z_k$. Let f be a measureable function on Ω and let μ_b be a Dirichlet measure on the standard simplex E in R^{k-1}.Define

$$F(b,z) = \int_E f(u.z) d\,\mu_b(u) \tag{2.3}$$

F is the Dirichlet measure of f with variables
$z = (z_1, \dots\dots\dots, z_k)$ and parameters $b = (b_1, \dots\dots\dots . b_k)$.
Here

$$u.z = \sum_{i=1}^{k} u_i z_i \text{ and } u_k = 1 - u_1 - \cdots \dots\dots . -u_{k-1.}$$

If $k = 1$, define $F(b,z) = f(z)$.

Fractional Derivative [9, p.181]

The theory of fractional derivative with respect to an arbitrary function has been used by Erdelyi [8]. The general definition for the fractional derivative of order α found in the literature on the "Riemann-Liouville integral" is

$$D_z^{\alpha} F(z) = \frac{1}{\Gamma(-\alpha)} \int_0^z F(t)(z-t)^{-\alpha-1} dt \tag{2.4}$$

Where $Re(\alpha) < 0$ and $F(x)$ is the form of $x^p f(x)$, where $f(x)$ is analytic at $x = 0$.

The New Generalization of Generalized M-Series

Here , first the notation and the definition of the New Generalized M-series, introduced by Ahmad Faraj , Tariq Salim , Safaa Sadek, Jamal Ismai [10] has been given as

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(a_1 \dots, a_p; b_1, \dots, b_q; z) = \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{z}),$$

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{z}) = \sum_{\mathrm{k}=0}^{\infty} \frac{(\mathrm{a}_1)_{km} \dots (a_p)_{km}}{(\mathrm{b}_1)_{kn} \dots (b_q)_{kn}} \; \frac{\mathrm{z}^{\mathrm{k}}}{\Gamma(\alpha k + \beta)}$$

…(2.8)

Here $\alpha, \beta \in C, \operatorname{Re}(\alpha) > 0, \ \operatorname{Re}(\beta) > 0$,$(\mathrm{a_j})_{\mathrm{km}}$, $(\mathrm{b_j})_{\mathrm{kn}}$ are the pochammer symbols and m,n are non-negative real numbers.

Average of $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{z})$ (from [16]):

let μ^b be a Dirichlet measure on the standard simplex E in R^{k-1}; k≥ 2. For every $z \in C^k$

$$S(b,z) = \int_E \underset{p,q;m,n}{\overset{\alpha,\beta}{M}} (u.z)\, d\mu_b\ (u) \tag{2.5}$$

If $k = 1, S = (b,z) = \underset{p,q;m,n}{\overset{\alpha,\beta}{M}} (u.z)$.

Triple averages of functions of one variable (from [1, 2])

let z be species with complex elements z_{ijk}. Let $u = (u_1, \dots\dots\dots u_l)$, $v = (v_1, \dots\dots\dots\dots v_m)$ and $w = (w_1, \dots\dots\dots\dots w_n)$ be an ordered l-tuple, m-tuple and n-tuple of real non-negative weights $\sum u_i = 1$, $\sum v_j = 1$, and $\sum w_k = 1$ respectively.
We define

$$u.z.v.w=\sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n}u_i z_{ijk} v_j w_k \tag{2.6}$$

If z_{ijk} is regarded as a point of the complex plane, all these convex combinations are points in the convex hull of $(z_{11},\dots\dots\dots z_{kx})$, denote by $H(z)$.

Let $\mu=(\mu_1,\dots\dots\dots.\mu_k)$ be an ordered l –tuple of complex numbers with positive real part (Re (μ) > 0) and similarly for($\alpha=\alpha_1$.................α_m) and β=(β_1,...........β_n). Then we define $dm_\mu(u), dm_\alpha(v)$ and $dm_\beta(w)$ as (2.2.1).

Let f be the holomorphic on a domain D in the complex plane. If $Re(\mu)>0, Re(\alpha), Re(\beta)>0 and H(z)\subset D$, we define

$$F(\mu,z,\alpha,\beta)=\iint\int \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(u,z,v,w)\,dm_\mu(u)\,dm_\alpha(v)\,dm_\beta(w) \tag{2.7}$$

$$R_t(\mu,z,\alpha,\beta)=\iint (u,z,v,w)^t dm_\mu(u)\,dm_\alpha(v)\,dm_\beta(w) \tag{2.8}$$

$$S(\mu,z,\alpha,\beta)=\iint (e)^{u.z.v.w}\,dm_\mu(u)\,dm_\alpha(v)\,dm_\beta(w) \tag{2.9}$$

MAIN RESULTS AND PROOF

Theorem: Following equivalence relation for Double Dirichlet Average is established for ($k=x=2$) of $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(u,z,v)$

$$\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mu,\mu';x;\rho,\rho')=\frac{\Gamma(\rho+\rho')}{\Gamma\rho}(x-y)^{1-\rho-\rho'}D_{x-y}^{-\rho'}\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(\mathrm{x})(x-y)^{\rho-1} \tag{3.1}$$

Proof:

Let us consider the triple average for ($l=m=n=2$ of $\underset{p,q;m,n}{\overset{\alpha,\beta}{M}}$ (u.z.v.w)

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta')=\int_0^1\int_0^1 \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(u.z.v.w)dm_{\mu,\mu'}(u)dm_{\alpha,\alpha'}(v)dm_{\beta,\beta'}(w)$$

$$=\sum_{\mathrm{k}=0}^{\infty}\frac{(\mathrm{a}_1)_{km}\dots(a_p)_{km}}{(\mathrm{b}_1)_{kn}\dots(b_q)_{kn}}\;\frac{1}{\Gamma(\alpha k+\beta)}\int_0^1\int_0^1[u.z.v.w]^k dm_{\mu,\mu'}(u)dm_{\alpha,\alpha'}(v)dm_{\beta,\beta'}(w) \tag{3.2}$$

$Re(\mu)=0, Re(\mu')=0, Re(\alpha)>0, Re(\alpha')>0, Re(\beta)>0, Re(\beta')>0$ and

$$u.z.v.w=\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}u_i z_{ijk} v_j w_k=\sum_{i=1}^{2}\sum_{j=1}^{2}[u_i v_j(z_{ij1}w_1+z_{ij2}w_2)]$$

$$u.z.v.w=\sum_{i=1}^{2}[u_i(v_1 z_{i11}w_1+v_1 z_{i12}w_2+v_2 z_{i21}w_1+v_2 z_{i22}w_2)]$$

$$u.z.v.w=[u_1v_1z_{111}w_1+u_1v_1z_{112}w_2+u_1v_2z_{121}w_1+u_1v_2z_{122}w_2+u_2v_1z_{211}w_1 \\ +u_2v_1z_{212}w_2+u_2v_2z_{221}w_1+u_2v_2z_{222}w_2]$$

let in first species $z_{111}=a, z_{112}=b, z_{121}=c, z_{122}=d$ and second species
$z_{211}=e, z_{212}=f, z_{221}=g, z_{222}=h$

and $\begin{cases} u_1=u, & u_2=1-u \\ v_1=v, & v_2=1-v \\ w_1=w, & w_2=1-w \end{cases}$

such that

$$u.z.v.w = [uvw(a-b-c+d-e+f+g-h) + uv(b-d-f+h) \\ +vw(e-f-g+h) + wu(c-d-g+h) + u(d-h) + v(f-h) + w(g-h) + h]$$

$$dm_{\mu,\mu'}(u) = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'} u^{\mu-1}(1-u)^{\mu'-1}du$$
$$dm_{\alpha,\alpha'}(v) = \frac{\Gamma(\alpha+\alpha')}{\Gamma\alpha\Gamma\alpha'} v^{\alpha-1}(1-v)^{\alpha'-1}dv$$
$$dm_{\beta,\beta'}(w) = \frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'} w^{\beta-1}(1-w)^{\beta'-1}dw$$

Putting these values in (3.2), we have,

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'} \\ \times \sum_{k=0}^{\infty} \frac{(a_1)_{km}\ldots(a_p)_{km}}{(b_1)_{kn}\ldots(b_q)_{kn}} \frac{1}{\Gamma(\alpha k+\beta)} \int_0^1\int_0^1\int_0^1 [uvw(a-b-c+d-e+f+g-h) + uv(b-d-f+h) \\ +vw(e-f-g+h) + wu(c-d-g+h) + u(d-h) + v(f-h) + w(g-h) + h]^{(n+\gamma)\alpha-\beta-1} \\ \times u^{\mu-1}(1-u)^{\mu'-1}du\; v^{\alpha-1}(1-v)^{\alpha'-1}dv\; w^{\beta-1}(1-w)^{\beta'-1}dw \tag{3.3}$$

In order to obtained the fractional derivative equivalent to the above integral.

Case-1:

If $a = x, e = y, b = c = d = f = g = h = 0$ then we have

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta') = \frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'}\frac{\Gamma(\rho+\rho')}{\Gamma\rho\Gamma\rho'}\frac{\Gamma(\beta+\beta')}{\Gamma\beta\Gamma\beta'} \\ \times \sum_{k=0}^{\infty} \frac{(a_1)_{km}\ldots(a_p)_{km}}{(b_1)_{kn}\ldots(b_q)_{kn}} \frac{1}{\Gamma(\alpha k+\beta)} \int_0^1\int_0^1\int_0^1 [uvw(x-y) + vwy]^k\, u^{\mu-1}(1-u)^{\mu'-1}\, v^{\alpha-1}(1 \\ -v)^{\alpha'-1}\, w^{\beta-1}(1-w)^{\beta'-1}dudvdw$$

Using the definition of beta function and due to suitable adjustment we arrive at

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma\mu\,\Gamma\mu'} \\ \times \sum_{k=0}^{\infty} \frac{(a_1)_{km}\ldots(a_p)_{km}}{(b_1)_{kn}\ldots(b_q)_{kn}} \frac{1}{\Gamma(\alpha k+\beta)} \int_0^1 [ux + (1-u)y]^k\, u^{\mu-1}(1-u)^{\mu'-1}du$$

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n} S(\mu,\mu';x,y) \tag{3.4}$$

By using the definition of fractional derivative we get,

$$S(\mu,\mu';z;\alpha,\alpha',\beta,\beta') = \frac{(\alpha)_n(\beta)_n}{(\alpha+\alpha')_n(\beta+\beta')_n}\frac{\Gamma(\mu+\mu')}{\Gamma\mu}(x-y)^{1-\mu-\mu'} D_{x-y}^{-\mu'} \underset{p,q;m,n}{\overset{\alpha,\beta}{M}}(x)(x-y)^{\mu-1} \tag{3.5}$$

This is complete proof of (3.1).

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